July 25, 2007

Problem 1. Real numbers $a_1, a_2, \dots, a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

and let

$$d = \max\{d_i : 1 \le i \le n\}.$$

(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that equality holds in $(*)$.

Problem 2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Problem 3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points

Language: English

July 26, 2007

Problem 4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 5. Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

Problem 6. Let n be a positive integer. Consider

$$S = \{(x, y, z) \ : \ x, y, z \in \{0, 1, \ldots, n\},\ x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points