T–1. Problem

Prove that for all positive real numbers a, b, c such that $abc = 1$ the following inequality holds:

$$\frac{a}{2b+c^2} + \frac{b}{2c+a^2} + \frac{c}{2a+b^2} \leqslant \frac{a^2+b^2+c^2}{3}.$$

T–2. Problem

Determine all functions $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ such that

$$f(x^2yf(x)) + f(1) = x^2f(x) + f(y)$$

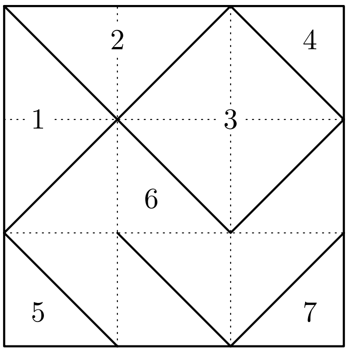
holds for all nonzero real numbers x and y.

T–3. Problem

There are n students standing in line in positions 1 to n. While the teacher looks away, some students change their positions. When the teacher looks back, they are standing in line again. If a student who was initially in position i is now in position j, we say the student moved for $|i-j|$ steps. Determine the maximal sum of steps of all students that they can achieve.

T–4. Problem

Let N be a positive integer. In each of the N^2 unit squares of an $N \times N$ board, one of the two diagonals is drawn. The drawn diagonals divide the $N \times N$ board into K regions. For each N, determine the smallest and the largest possible values of K.



Example with $N = 3$, $K = 7$

T–5. Problem

Let ABC be an acute triangle with $AB < AC$. Prove that there exists a point D with the following property: whenever two distinct points X and Y lie in the interior of ABC such that the points B, C, X, and Y lie on a circle and

$$\angle AXB - \angle ACB = \angle CYA - \angle CBA$$

holds, the line XY passes through D.

T–6. Problem

Let I be the incentre of triangle ABC and let the angle bisector AI intersect the side BC at D. Suppose that the point P lies on the line BC and satisfies $PI = PD$. Further, let J be the point obtained by reflecting I over the perpendicular bisector of BC, and let Q be the other intersection of the circumcircles of the triangles ABC and APD. Prove that $\angle BAQ = \angle CAJ$.

T–7. Problem

Find all pairs of positive integers (a, b) such that

$$a! + b! = a^b + b^a.$$

T–8. Problem

Let $n \geqslant 2$ be an integer. Determine the number of positive integers m such that $m \leqslant n$ and $m^2 + 1$ is divisible by n.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.