Problem T–1

Consider the two infinite sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ of real numbers such that $a_0 = 0$, $b_0 = 0$ and

$$a_{k+1} = b_k, \qquad b_{k+1} = \frac{a_k b_k + a_k + 1}{b_k + 1}$$

for each integer $k \geq 0$. Prove that $a_{2024} + b_{2024} \geq 88$.

Problem T–2

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$yf(x+1) = f(x + y - f(x)) + f(x)f(f(y))$$

for all $x, y \in \mathbb{R}$.

Problem T–3

There are 2024 mathematicians sitting in a row next to the river Tisza. Each of them is working on exactly one research topic, and if two mathematicians are working on the same topic, everyone sitting between them is also working on it.

Marvin is trying to figure out for each pair of mathematicians whether they are working on the same topic. He is allowed to ask each mathematician the following question: “How many of these 2024 mathematicians are working on your topic?” He asks the questions one by one, so he knows all previous answers before he asks the next one.

Determine the smallest positive integer k such that Marvin can always accomplish his goal with at most k questions.

Problem T–4

A finite sequence $x_1, x_2, \ldots, x_r$ of positive integers is a *palindrome* if $x_i = x_{r+1-i}$ for all integers $1 \leq i \leq r$.

Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. For a positive integer $j \geq 2$, denote by $a[j]$ the finite subsequence $a_1, a_2, \ldots, a_{j-1}$. Suppose that there exists a strictly increasing infinite sequence $b_1, b_2, \ldots$ of positive integers such that for every positive integer n, the subsequence $a[b_n]$ is a palindrome and $b_{n+2} \leq b_{n+1} + b_n$. Prove that there exists a positive integer T such that $a_i = a_{i+T}$ for every positive integer i.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let ABC be a triangle with $\angle BAC = 60^\circ$. Let D be a point on the line AC such that $AB = AD$ and A lies between C and D. Suppose that there are two points $E \neq F$ on the circumcircle of the triangle DBC such that $AE = AF = BC$. Prove that the line EF passes through the circumcenter of ABC.

Problem T–6

Let ABC be an acute triangle. Let M be the midpoint of the segment BC. Let I, J, K be the incenters of triangles ABC, ABM, ACM, respectively. Let P, Q be points on the lines MK, MJ, respectively, such that $\angle AJP = \angle ABC$ and $\angle AKQ = \angle BCA$. Let R be the intersection of the lines CP and BQ. Prove that the lines IR and BC are perpendicular.

Problem T–7

Define *glueing* of positive integers as writing their base ten representations one after another and interpreting the result as the base ten representation of a single positive integer.

Find all positive integers k for which there exists an integer N_k with the following property: for all $n \geq N_k$, we can glue the numbers $1, 2, \ldots, n$ in some order so that the result is a number divisible by k.

Remark. The base ten representation of a positive integer never starts with zero.

Example. Glueing $15, 14, 7$ in this order makes 15147.

Problem T–8

Let k be a positive integer and $a_1, a_2, \ldots$ be an infinite sequence of positive integers such that

$$a_i a_{i+1} \mid k - a_i^2$$

for all integers $i \geq 1$. Prove that there exists a positive integer M such that $a_n = a_{n+1}$ for all integers $n \geq M$.